

# Weak solutions of the Landau–Lifshitz–Bloch equation <sup>\*</sup>

Kim Ngan Le <sup>†</sup>

May 18, 2016

## Abstract

The Landau–Lifshitz–Bloch (LLB) equation is a formulation of dynamic micromagnetics valid at all temperatures, treating both the transverse and longitudinal relaxation components important for high-temperature applications. We study LLB equation in case the temperature raised higher than the Curie temperature. The existence of weak solution is showed and its regularity properties are also discussed. In this way, we lay foundations for the rigorous theory of LLB equation that is currently not available.

**Key words:** Landau–Lifshitz–Bloch, quasilinear parabolic equation, ferromagnetism

**AMS subject classifications:** 82D40, 35K59, 35R15

## 1 Introduction

Micromagnetic modeling has proved itself as a widely used tool, complimentary in many respects to experimental measurements. The Landau–Lifshitz–Gilbert (LLG) equation [21, 16] provides a basis for this modeling, especially where the dynamical behaviour is concerned. According to this theory, at temperatures below the critical (so-called Curie) temperature, the magnetization  $\mathbf{m}(t, \mathbf{x}) \in \mathbb{S}^2$ , where  $\mathbb{S}^2$  is the unit sphere in  $\mathbb{R}^3$ , for  $t > 0$  and  $\mathbf{x} \in D \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$ , satisfies the following LLG equation

$$\frac{\partial \mathbf{m}}{\partial t} = \lambda_1 \mathbf{m} \times \mathbf{H}_{\text{eff}} - \lambda_2 \mathbf{m} \times (\mathbf{m} \times \mathbf{H}_{\text{eff}}), \quad (1.1)$$

where  $\times$  is the vector cross product in  $\mathbb{R}^3$  and  $\mathbf{H}_{\text{eff}}$  is the so-called effective field.

---

<sup>\*</sup>This work was supported by the Australian Research Council grant DP140101193.

<sup>†</sup>School of Mathematics and Statistics, The University of New South Wales, Sydney 2052, Australia Email: [n.le-kim@unsw.edu.au](mailto:n.le-kim@unsw.edu.au)

However, for high temperatures the model must be replaced by a more thermodynamically consistent approach such as the Landau–Lifshitz–Bloch (LLB) equation [14, 15]. The LLB equation essentially interpolates between the LLG equation at low temperatures and the Ginzburg-Landau theory of phase transitions. It is valid not only below but also above the Curie temperature  $T_c$ . An important property of the LLB equation is that the magnetization magnitude is no longer conserved but is a dynamical variable [15, 11]. The spin polarization  $\mathbf{u}(t, \mathbf{x}) \in \mathbb{R}^3$ , ( $\mathbf{u} = \mathbf{m}/m_s^0$ ,  $\mathbf{m}$  is magnetization and  $m_s^0$  is the saturation magnetization value at  $T = 0$ ), for  $t > 0$  and  $\mathbf{x} \in D \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$ , satisfies the following LLB equation

$$\frac{\partial \mathbf{u}}{\partial t} = \gamma \mathbf{u} \times \mathbf{H}_{\text{eff}} + L_1 \frac{1}{|\mathbf{u}|^2} (\mathbf{u} \cdot \mathbf{H}_{\text{eff}}) \mathbf{u} - L_2 \frac{1}{|\mathbf{u}|^2} \mathbf{u} \times (\mathbf{u} \times \mathbf{H}_{\text{eff}}). \quad (1.2)$$

Here,  $|\cdot|$  is the Euclidean norm in  $\mathbb{R}^3$ ,  $\gamma > 0$  is the gyromagnetic ratio, and  $L_1$  and  $L_2$  are the longitudinal and transverse damping parameters, respectively.

LLB micromagnetics has become a real alternative to LLG micromagnetics for temperatures which are close to the Curie temperature ( $T \gtrsim \frac{3}{4}T_c$ ). This is realistic for some novel exciting phenomena, such as light-induced demagnetization with powerfull femtosecond (fs) lasers [2]. During this process the electronic temperature is normally raised higher than  $T_c$ . Micromagnetics based on the LLG equation cannot work under these circumstances while micromagnetics based on the LLB equation has proved to describe correctly the observed fs magnetization dynamics.

In this paper, we consider a deterministic form of a ferromagnetic LLB equation, in which the temperature  $T$  is raised higher than  $T_c$ , and as a consequence the longitudinal  $L_1$  and transverse  $L_2$  damping parameters are equal. The effective field  $\mathbf{H}_{\text{eff}}$  is given by

$$\mathbf{H}_{\text{eff}} = \Delta \mathbf{u} - \frac{1}{\chi_{\parallel}} \left( 1 + \frac{3}{5} \frac{T}{T - T_c} |\mathbf{u}|^2 \right) \mathbf{u},$$

where  $\chi_{\parallel}$  is the longitudinal susceptibility.

By using the vector triple product identity  $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ , we get

$$\mathbf{u} \times (\mathbf{u} \times \mathbf{H}_{\text{eff}}) = (\mathbf{u} \cdot \mathbf{H}_{\text{eff}}) \mathbf{u} - |\mathbf{u}|^2 \mathbf{H}_{\text{eff}},$$

and from property  $L_1 = L_2 =: \kappa_1$ , we can rewrite (1.2) as follows

$$\frac{\partial \mathbf{u}}{\partial t} = \kappa_1 \Delta \mathbf{u} + \gamma \mathbf{u} \times \Delta \mathbf{u} - \kappa_2 (1 + \mu |\mathbf{u}|^2) \mathbf{u}, \quad \text{with } \kappa_2 := \frac{\kappa_1}{\chi_{\parallel}}, \quad \mu := \frac{3T}{5(T - T_c)}. \quad (1.3)$$

So the LLB equation we are going to study in this paper is equation (1.3) with real positive coefficients  $\kappa_1, \kappa_2, \gamma, \mu$ , initial data  $\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x})$  and subject to homogeneous Neumann boundary conditions.

Various results on existence of global weak solutions of the LLG equation (1.1) are proved in [8, 1]. More complete lists can be found in [9, 18, 20]. Furthermore, there is also some research about the weak solution of its stochastic version (i.e., the effective

field is perturbed by a Gaussian noise), such as in [6, 4]. It should be mentioned that the proof of existence in [3, 5, 17] is a constructive proof, namely an approximate solution can be computed.

To the best of our knowledge the analysis of the LLB equation is an open problem at present. In this paper, we introduce a definition of weak solutions of the LLB equation. By introducing the Faedo–Galerkin approximations and using the method of compactness, we prove the existence of weak solutions for the LLB equation.

This paper is organized as follows. In Section 2 we introduce the notations and formulate the main result (Theorem 2.2) on the existence of the weak solution of (1.3) as well as some regularity properties. In Section 3 we introduce the Faedo–Galerkin approximations and prove for them some uniform bounds in various norms. In Section 4, we use the method of compactness to show the existence of a weak solution and prove the main theorem. Finally, in the Appendix we collect, for the reader's convenience, some facts scattered in the literature that are used in the course of the proof.

## 2 Notation and the formulation of the main result

Before presenting the definition of a weak solution to the LLB equation (1.3), it is necessary to introduce some function spaces.

The function spaces  $\mathbb{H}^1(D, \mathbb{R}^3) =: \mathbb{H}^1$  are defined as follows:

$$\mathbb{H}^1(D, \mathbb{R}^3) = \left\{ \mathbf{u} \in \mathbb{L}^2(D, \mathbb{R}^3) : \frac{\partial \mathbf{u}}{\partial x_i} \in \mathbb{L}^2(D, \mathbb{R}^3) \text{ for } i = 1, 2, 3. \right\}.$$

Here,  $\mathbb{L}^p(D, \mathbb{R}^3) =: \mathbb{L}^p$  with  $p > 0$  is the usual space of  $p^{\text{th}}$ -power Lebesgue integrable functions defined on  $D$  and taking values in  $\mathbb{R}^3$ . Throughout this paper, we denote a scalar product in a Hilbert space  $H$  by  $\langle \cdot, \cdot \rangle_H$  and its associated norm by  $\| \cdot \|_H$ . The dual brackets between a space  $X$  and its dual  $X^*$  will be denoted  ${}_X \langle \cdot, \cdot \rangle_{X^*}$ .

**Definition 2.1.** *Given  $T > 0$ , a weak solution  $\mathbf{u} : [0, T] \rightarrow \mathbb{H}^1 \cap \mathbb{L}^4$  to (1.3) satisfies*

$$\begin{aligned} \langle \mathbf{u}(t), \boldsymbol{\phi} \rangle_{\mathbb{L}^2} = & \langle \mathbf{u}_0, \boldsymbol{\phi} \rangle_{\mathbb{L}^2} - \kappa_1 \int_0^t \langle \nabla \mathbf{u}(s), \nabla \boldsymbol{\phi} \rangle_{\mathbb{L}^2} ds - \gamma \int_0^t \langle \mathbf{u}(s) \times \nabla \mathbf{u}(s), \nabla \boldsymbol{\phi} \rangle_{\mathbb{L}^2} ds \\ & - \kappa_2 \int_0^t \langle (1 + \mu |\mathbf{u}|^2(s)) \mathbf{u}(s), \boldsymbol{\phi} \rangle_{\mathbb{L}^2} ds, \end{aligned} \quad (2.1)$$

for every  $\boldsymbol{\phi} \in \mathbb{C}_0^\infty(D)$  and  $t \in [0, T]$ .

Now we can formulate the main result of this paper.

**Theorem 2.2.** *Let  $D \subset \mathbb{R}^d$  be an open bounded domain with  $C^m$  extension property and assume that  $d < 2m$ . For  $T > 0$  and for the initial data  $\mathbf{u}_0 \in \mathbb{H}^1$ , there exists a weak solution of (1.3) such that*

(a) for every  $t \in [0, T]$ ,

$$\begin{aligned} \mathbf{u}(t) = & \mathbf{u}_0 + \kappa_1 \int_0^t \Delta \mathbf{u}(s) ds + \gamma \int_0^t \mathbf{u}(s) \times \Delta \mathbf{u}(s) ds \\ & - \kappa_2 \int_0^t (1 + \mu |\mathbf{u}|^2(s)) \mathbf{u}(s) ds \quad \text{in } \mathbb{L}^{3/2}, \end{aligned} \quad (2.2)$$

(b) for every  $\alpha \in (0, \frac{1}{4}]$ ,  $\mathbf{u} \in C^\alpha([0, T], \mathbb{L}^{3/2})$ ,

(c)  $\sup_{t \in [0, T]} \|\mathbf{u}(t, \cdot)\|_{\mathbb{L}^2} < \infty$ .

**Remark 2.3.** The notation  $\Delta \mathbf{u}$  and  $\mathbf{u} \times \Delta \mathbf{u}$  will be defined in the Notations 4.1–4.2.

### 3 Faedo-Galerkin Approximation

Let  $A = -\Delta$  be the negative Laplace operator. From [10, Theorem 1, p. 335], there exists an orthonormal basis  $\{\mathbf{e}_i\}_{i=1}^\infty$  of  $\mathbb{L}^2$ , consisting of eigenvectors for operator  $A$ , such that  $\mathbf{e}_i \in \mathbb{C}^m(D) \cap \mathbb{L}^\infty$  for all  $i = 1, 2, \dots$  and

$$-\Delta \mathbf{e}_i = \lambda_i \mathbf{e}_i, \quad \mathbf{e}_i = 0 \text{ on } \partial D,$$

where  $\lambda_i > 0$  for  $i = 1, 2, \dots$  are eigenvalues of  $A$ . Let  $S_n := \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  and  $\Pi_n$  be the orthogonal projection from  $\mathbb{L}^2$  onto  $S_n$ , defined by: for  $\mathbf{v} \in \mathbb{L}^2$

$$\langle \Pi_n \mathbf{v}, \boldsymbol{\phi} \rangle_{\mathbb{L}^2} = \langle \mathbf{v}, \boldsymbol{\phi} \rangle_{\mathbb{L}^2}, \quad \forall \boldsymbol{\phi} \in S_n. \quad (3.1)$$

By taking  $\boldsymbol{\phi} = \Pi_n \mathbf{v}$  in the above equation, we obtain an upper bound for the projection operator  $\Pi_n$  in  $\mathbb{L}^2$ ,

$$\|\Pi_n \mathbf{v}\|_{\mathbb{L}^2} \leq \|\mathbf{v}\|_{\mathbb{L}^2} \quad \forall \mathbf{v} \in S_n. \quad (3.2)$$

We note that  $\Pi_n$  is a self-adjoint operator on  $\mathbb{L}^2$ , indeed, from (3.1), for  $\mathbf{v}, \mathbf{w} \in \mathbb{L}^2$  there holds

$$\langle \mathbf{w}, \Pi_n \mathbf{v} \rangle_{\mathbb{L}^2} = \langle \Pi_n \mathbf{v}, \Pi_n \mathbf{w} \rangle_{\mathbb{L}^2} = \langle \mathbf{v}, \Pi_n \mathbf{w} \rangle_{\mathbb{L}^2}.$$

We are now looking for approximate solution  $\mathbf{u}_n(\cdot, t) \in S_n := \text{span}\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$  of equation (1.3) satisfying

$$\frac{\partial \mathbf{u}_n}{\partial t} - \kappa_1 \Delta \mathbf{u}_n - \gamma \Pi_n (\mathbf{u}_n \times \Delta \mathbf{u}_n) + \kappa_2 \Pi_n ((1 + \mu |\mathbf{u}_n|^2) \mathbf{u}_n) = 0, \quad (3.3)$$

with  $\mathbf{u}_n(\cdot, 0) = \mathbf{u}_{0n}$ , where  $\mathbf{u}_{0n} \in S_n$  is an approximation of  $\mathbf{u}_0$ . Since equation (3.3) is equivalent to an ordinary differential equation in  $\mathbb{R}^n$ , the existence of a local solution to (3.3) is a consequence of the following lemma.

**Lemma 3.1.** For  $n \in \mathbb{N}$ , define the maps:

$$\begin{aligned} F_n^1 : S_n \ni \mathbf{v} &\mapsto \Delta \mathbf{v} \in S_n, \\ F_n^2 : S_n \ni \mathbf{v} &\mapsto \Pi_n(\mathbf{v} \times \Delta \mathbf{v}) \in S_n, \\ F_n^3 : S_n \ni \mathbf{v} &\mapsto \Pi_n((1 + \mu|\mathbf{v}|^2)\mathbf{v}) \in S_n. \end{aligned}$$

Then  $F_n^1$  is globally Lipschitz and  $F_n^2, F_n^3$  are locally Lipschitz.

*Proof.* For any  $\mathbf{v} \in S_n$  we have

$$\mathbf{v} = \sum_{i=1}^n \langle \mathbf{v}, \mathbf{e}_i \rangle_{\mathbb{L}^2} \mathbf{e}_i \quad \text{and} \quad -\Delta \mathbf{v} = \sum_{i=1}^n \lambda_i \langle \mathbf{v}, \mathbf{e}_i \rangle_{\mathbb{L}^2} \mathbf{e}_i.$$

By using the triangle inequality, the orthonormal property of  $\{\mathbf{e}_i\}_{i=1}^n$  and Hölder's inequality, for any  $\mathbf{u}, \mathbf{v} \in S_n$  we obtain

$$\begin{aligned} \|F_n^1(\mathbf{u}) - F_n^1(\mathbf{v})\|_{\mathbb{L}^2} &= \|\Delta \mathbf{u} - \Delta \mathbf{v}\|_{\mathbb{L}^2} = \left\| \sum_{i=1}^n \lambda_i \langle \mathbf{u} - \mathbf{v}, \mathbf{e}_i \rangle_{\mathbb{L}^2} \mathbf{e}_i \right\|_{\mathbb{L}^2} \\ &\leq \sum_{i=1}^n \lambda_i |\langle \mathbf{u} - \mathbf{v}, \mathbf{e}_i \rangle_{\mathbb{L}^2}| \leq \left( \sum_{i=1}^n \lambda_i \right) \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2}, \end{aligned}$$

then the globally Lipschitz property of  $F_n^1$  follows immediately.

From (3.2) and the triangle inequality, there holds

$$\begin{aligned} \|F_n^2(\mathbf{u}) - F_n^2(\mathbf{v})\|_{\mathbb{L}^2} &= \|\Pi_n(\mathbf{u} \times \Delta \mathbf{u} - \mathbf{v} \times \Delta \mathbf{v})\|_{\mathbb{L}^2} \leq \|\mathbf{u} \times \Delta \mathbf{u} - \mathbf{v} \times \Delta \mathbf{v}\|_{\mathbb{L}^2} \\ &\leq \|\mathbf{u} \times (\Delta \mathbf{u} - \Delta \mathbf{v})\|_{\mathbb{L}^2} + \|(\mathbf{u} - \mathbf{v}) \times \Delta \mathbf{v}\|_{\mathbb{L}^2} \\ &\leq \|\mathbf{u}\|_{\mathbb{L}^\infty} \|F_n^1(\mathbf{u}) - F_n^1(\mathbf{v})\|_{\mathbb{L}^2} + \|(\mathbf{u} - \mathbf{v})\|_{\mathbb{L}^2} \|\Delta \mathbf{v}\|_{\mathbb{L}^\infty}. \end{aligned}$$

Since  $F_n^1$  is globally Lipschitz and the fact that all norms are equivalent in the finite dimensional space  $S_n$ ,  $F_n^2$  is locally Lipschitz.

Similarly, the local Lipschitz property of  $F_n^3$  follows from the estimate,

$$\begin{aligned} \|F_n^3(\mathbf{u}) - F_n^3(\mathbf{v})\|_{\mathbb{L}^2} &\leq \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2} + \mu \|\Pi_n(|\mathbf{u}|^2 \mathbf{u} - |\mathbf{v}|^2 \mathbf{v})\|_{\mathbb{L}^2} \\ &\leq \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2} + \mu \| |\mathbf{u}|^2 \mathbf{u} - |\mathbf{v}|^2 \mathbf{v} \|_{\mathbb{L}^2} \\ &\leq \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2} + \mu \| |\mathbf{u}|^2 (\mathbf{u} - \mathbf{v}) \|_{\mathbb{L}^2} + \mu \| (\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \mathbf{v} \|_{\mathbb{L}^2} \\ &\leq (1 + \mu \| |\mathbf{u}|^2 \|_{\mathbb{L}^\infty} + \mu \|\mathbf{u} + \mathbf{v}\|_{\mathbb{L}^\infty} \|\mathbf{v}\|_{\mathbb{L}^\infty}) \|\mathbf{u} - \mathbf{v}\|_{\mathbb{L}^2}, \end{aligned}$$

which complete the proof of this lemma.  $\square$

We now proceed to priori estimates on the approximate solution  $\mathbf{u}_n$ .

**Lemma 3.2.** *For each  $n = 1, 2, \dots$  and every  $t \in [0, T]$ ,*

$$\|\mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + 2\kappa_1 \int_0^t \|\nabla \mathbf{u}_n(s)\|_{\mathbb{L}^2}^2 ds + 2\kappa_2 \int_0^t (\|\mathbf{u}_n(s)\|_{\mathbb{L}^2}^2 + \mu \|\mathbf{u}_n(s)\|_{\mathbb{L}^4}^4) ds \leq \|\mathbf{u}_n(0)\|_{\mathbb{L}^2}^2,$$

and

$$\|\nabla \mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + 2\kappa_1 \int_0^t \|\Delta \mathbf{u}_n(s)\|_{\mathbb{L}^2}^2 ds \leq \|\nabla \mathbf{u}_n(0)\|_{\mathbb{L}^2}^2.$$

*Proof.* Taking the inner product of both sides of (3.3) with  $\mathbf{u}_n(t) \in S_n$ , integrating by parts with respect to  $\mathbf{x}$ , and using  $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = 0$  and the fact that  $\Pi_n$  is self-adjoint, we obtain

$$\frac{1}{2} \frac{\partial}{\partial t} \|\mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + \kappa_1 \|\nabla \mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + \kappa_2 \langle (1 + \mu |\mathbf{u}_n|^2) \mathbf{u}_n, \mathbf{u}_n(t) \rangle_{\mathbb{L}^2} = 0.$$

The first result follows by integrating both sides of the above equation with respect to  $t$ .

In a similar fashion, we next take the inner product of both sides of (3.3) with  $\Delta \mathbf{u}_n(t) \in S_n$ , and then integrate by parts with respect to  $\mathbf{x}$  to arrive at

$$\begin{aligned} \frac{1}{2} \frac{\partial}{\partial t} \|\nabla \mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + \kappa_1 \|\Delta \mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + \kappa_2 \langle (1 + \mu |\mathbf{u}_n|^2) \nabla \mathbf{u}_n, \nabla \mathbf{u}_n(t) \rangle_{\mathbb{L}^2} \\ + \kappa_2 \langle 2\mu (\mathbf{u}_n \cdot \nabla \mathbf{u}_n) \mathbf{u}_n, \nabla \mathbf{u}_n(t) \rangle_{\mathbb{L}^2} = 0 \end{aligned}$$

Integrating both sides with respect to  $t$ , we obtain

$$\begin{aligned} \|\nabla \mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + 2\kappa_1 \int_0^t \|\Delta \mathbf{u}_n(s)\|_{\mathbb{L}^2}^2 ds + 2\kappa_2 \int_0^t \int_D (1 + \mu |\mathbf{u}_n|^2) (\nabla \mathbf{u}_n)^2 d\mathbf{x} ds \\ + 2\kappa_2 \mu \int_0^t \int_D (\mathbf{u}_n \cdot \nabla \mathbf{u}_n)^2 d\mathbf{x} ds = \|\nabla \mathbf{u}_n(0)\|_{\mathbb{L}^2}^2, \end{aligned}$$

and the second result follows immediately.  $\square$

The following upper bounds for  $\mathbf{u}_n \times \Delta \mathbf{u}_n$  and  $(1 + \mu |\mathbf{u}_n|^2) \mathbf{u}_n$  are a consequence of Lemma 3.2.

**Lemma 3.3.** *There exists a constant  $C$ , which does not depend on  $n = 1, 2, \dots$ , such that*

$$\int_0^T \|\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)\|_{\mathbb{L}^{3/2}}^2 dt \leq C \quad \text{and} \quad \int_0^T \|(1 + \mu |\mathbf{u}_n|^2(t)) \mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 dt \leq C.$$

*Proof.* By Hölder's inequality and the Sobolev imbedding of  $\mathbb{H}^1$  into  $\mathbb{L}^6$  [13] we have

$$\|\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)\|_{\mathbb{L}^{3/2}} \leq \|\mathbf{u}_n(t)\|_{\mathbb{L}^6} \|\Delta \mathbf{u}_n(t)\|_{\mathbb{L}^2} \leq C \|\mathbf{u}_n(t)\|_{\mathbb{H}^1} \|\Delta \mathbf{u}_n(t)\|_{\mathbb{L}^2}.$$

We use Lemma 3.2 to obtain the first result,

$$\int_0^T \|\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)\|_{\mathbb{L}^{3/2}}^2 dt \leq C \sup_{t \in [0, T]} \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^2 \int_0^T \|\Delta \mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 dt \leq C.$$

Similarly, from Lemma 3.2 and the Sobolev imbedding of  $\mathbb{H}^1$  into  $\mathbb{L}^6$ , we have

$$\|\mathbf{u}_n^3(t)\|_{\mathbb{L}^2}^2 = \|\mathbf{u}_n(t)\|_{\mathbb{L}^6}^6 \leq \|\mathbf{u}_n(t)\|_{\mathbb{H}^1}^6 \leq C, \quad (3.4)$$

so

$$\|(1 + \mu|\mathbf{u}_n|^2(t))\mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 \leq 2\|\mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 + 2\mu^2\|\mathbf{u}_n^3(t)\|_{\mathbb{L}^2}^2 \leq C,$$

and the second result follows immediately.  $\square$

Equation (3.3) can be written in the following way as an approximation of equation (1.3),

$$\begin{aligned} \mathbf{u}_n(t) &= \mathbf{u}_n(0) + \kappa_1 \int_0^t \Delta \mathbf{u}_n ds + \gamma \int_0^t \Pi_n(\mathbf{u}_n \times \Delta \mathbf{u}_n) ds - \kappa_2 \int_0^t \Pi_n((1 + \mu|\mathbf{u}_n|^2)\mathbf{u}_n) ds \\ &= \mathbf{u}_n(0) + \kappa_1 \mathbf{B}_{n,1}(t) + \gamma \mathbf{B}_{n,2}(t) + \kappa_2 \mathbf{B}_{n,3}(t). \end{aligned} \quad (3.5)$$

Before proving the uniform bound of  $\{\mathbf{u}_n\}$ , we define the following fractional power space [19, Definiton 1.4.7].

**Definition 3.4.** Put  $A_1 := I + A$ . For any real number  $\beta > 0$ , we define the Hilbert space

$$X^\beta := \{\phi \in \mathbb{L}^2 : \|A_1^\beta \phi\|_{\mathbb{L}^2} < \infty\},$$

where  $A_1^\beta \phi := \sum_{i=1}^\infty (1 + \lambda_i)^\beta \langle \phi, \mathbf{e}_i \rangle_{\mathbb{L}^2} \mathbf{e}_i$ , with the graph norm  $\|\cdot\|_{X^\beta} = \|A_1^\beta \cdot\|_{\mathbb{L}^2}$ . The dual space of  $X^\beta$  is denoted by  $X^{-\beta}$ .

The following lemma states an upper bound for the projection operator  $\Pi_n$  in  $X^{-\beta}$ .

**Lemma 3.5.** For any  $\beta > 0$  and  $\mathbf{v} \in \mathbb{L}^2$  there holds

$$\|\Pi_n \mathbf{v}\|_{X^{-\beta}} \leq \|\mathbf{v}\|_{X^{-\beta}}.$$

*Proof.* The proof of this lemma can be found in [7]; for the reader's convenience we recall the proof as follows.

For  $\mathbf{v} \in \mathbb{L}^2$ , by using (3.1) we obtain

$$\begin{aligned} \|\Pi_n \mathbf{v}\|_{X^{-\beta}} &= \sup_{\|\mathbf{w}\|_{X^\beta} \leq 1} |_{X^{-\beta}} \langle \Pi_n \mathbf{v}, \mathbf{w} \rangle_{X^\beta}| = \sup_{\|\mathbf{w}\|_{X^\beta} \leq 1} |\langle \Pi_n \mathbf{v}, \mathbf{w} \rangle_{\mathbb{L}^2}| \\ &= \sup_{\|\mathbf{w}\|_{X^\beta} \leq 1} |\langle \mathbf{v}, \Pi_n \mathbf{w} \rangle_{\mathbb{L}^2}|. \end{aligned} \quad (3.6)$$

Since

$$\|\Pi_n \mathbf{w}\|_{X^\beta}^2 = \sum_{i=1}^n (1 + \lambda_i)^{2\beta} \langle \mathbf{w}, \mathbf{e}_i \rangle_{\mathbb{L}^2}^2 \leq \sum_{i=1}^{\infty} (1 + \lambda_i)^{2\beta} \langle \mathbf{w}, \mathbf{e}_i \rangle_{\mathbb{L}^2}^2 = \|\mathbf{w}\|_{X^\beta}^2,$$

the set  $\{\mathbf{w} \in X^\beta : \|\mathbf{w}\|_{X^\beta} \leq 1\}$  is a subset of the set  $\{\mathbf{w} \in X^\beta : \|\Pi_n \mathbf{w}\|_{X^\beta} \leq 1\}$ . Hence, from (3.6) there holds

$$\|\Pi_n \mathbf{v}\|_{X^{-\beta}} \leq \sup_{\|\Pi_n \mathbf{w}\|_{X^\beta} \leq 1} |\langle \mathbf{v}, \Pi_n \mathbf{w} \rangle_{\mathbb{L}^2}| \leq \|\mathbf{v}\|_{X^{-\beta}},$$

which completes the proof of the lemma.  $\square$

We now prove a uniform bound for  $\{\mathbf{u}_n\}$  in  $H^1(0, T; X^{-\beta})$ .

**Lemma 3.6.** *Let  $D \subset \mathbb{R}^d$  be an open bounded domain with the  $C^m$  extension property. Given  $\beta > \frac{d}{6m}$ , there exists a constant  $C$ , which does not depend on  $n$  such that*

$$\|\mathbf{B}_{n,2}\|_{H^1(0,T;X^{-\beta})} \leq C, \quad (3.7)$$

$$\|\mathbf{B}_{n,3}\|_{H^1(0,T;\mathbb{L}^2)} \leq C, \quad (3.8)$$

and

$$\|\mathbf{u}_n\|_{H^1(0,T;X^{-\beta})} \leq C, \quad (3.9)$$

with  $\mathbf{B}_{n,2}$  and  $\mathbf{B}_{n,3}$  are defined in (3.5).

*Proof.* Since  $\beta > \frac{d}{6m}$ , by using Lemma 5.2 we infer that  $X^\beta$  is continuously embedded in  $\mathbb{L}^3$ . Thus we have the continuous imbedding

$$\mathbb{L}^{3/2} \hookrightarrow X^{-\beta}. \quad (3.10)$$

Proof of (3.7): By using Lemma 3.5, (3.10) and the first result of Lemma 3.3 we deduce

$$\begin{aligned} \int_0^T \left\| \frac{\partial}{\partial t} \mathbf{B}_{n,2}(t) \right\|_{X^{-\beta}}^2 dt &= \int_0^T \|\Pi_n(\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t))\|_{X^{-\beta}}^2 dt \\ &\leq C \int_0^T \|\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)\|_{X^{-\beta}}^2 dt \\ &\leq C \int_0^T \|\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)\|_{\mathbb{L}^{3/2}}^2 dt \leq C. \end{aligned} \quad (3.11)$$

In the same maner, we estimate  $\mathbf{B}_{n,2}$  in the norm of  $L^2(0, T; X^{-\beta})$  as follows. Since  $\mathbf{e}_i \in \mathbb{L}^\infty$  for  $i = 1, \dots, n$ , we see from Lemma 3.2 that

$$\int_0^t \int_D \left| (\mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s)) \cdot \mathbf{e}_i \right| dx ds \leq \|\mathbf{e}_i\|_{\mathbb{L}^\infty} \|\mathbf{u}_n\|_{L^2(0,T;\mathbb{L}^2)} \|\Delta \mathbf{u}_n\|_{L^2(0,T;\mathbb{L}^2)} < \infty,$$



and thus from Fubini's theorem there holds

$$\int_0^t \Pi_n(\mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s)) ds = \Pi_n \left( \int_0^t \mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s) ds \right). \quad (3.12)$$

By using (3.12), Lemma 3.5, (3.10) and Minkowski's inequality, we deduce

$$\begin{aligned} \int_0^T \|\mathbf{B}_{n,2}(t)\|_{X^{-\beta}}^2 dt &= \int_0^T \left\| \int_0^t \Pi_n(\mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s)) ds \right\|_{X^{-\beta}}^2 dt \\ &= \int_0^T \left\| \Pi_n \left( \int_0^t \mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s) ds \right) \right\|_{X^{-\beta}}^2 dt \\ &\leq \int_0^T \left\| \int_0^t \mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s) ds \right\|_{X^{-\beta}}^2 dt \\ &\leq \int_0^T \left\| \int_0^t \mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s) ds \right\|_{\mathbb{L}^{3/2}}^2 dt \\ &\leq \int_0^T \left( \int_0^t \|\mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s)\|_{\mathbb{L}^{3/2}} ds \right)^2 dt. \end{aligned}$$

Thus, it follows from Hölder's inequality and the first result of Lemma 3.3 that

$$\int_0^T \|\mathbf{B}_{n,2}(t)\|_{X^{-\beta}}^2 dt \leq \int_0^T t \int_0^t \|\mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s)\|_{\mathbb{L}^{3/2}}^2 ds dt \leq \int_0^T tC dt = CT^2. \quad (3.13)$$

The first result (3.7) follows immediately from (3.11) and (3.13).

Proof of (3.8): Using the same technique as in the proof of (3.7), we prove (3.8) as follows.

From (3.2) and the second result of Lemma 3.3, we deduce

$$\begin{aligned} \int_0^T \left\| \frac{\partial}{\partial t} \mathbf{B}_{n,3}(t) \right\|_{\mathbb{L}^2}^2 dt &= \int_0^T \|\Pi_n((1 + \mu|\mathbf{u}_n|^2(t))\mathbf{u}_n(t))\|_{\mathbb{L}^2}^2 dt \\ &\leq \int_0^T \|(1 + \mu|\mathbf{u}_n|^2(t))\mathbf{u}_n(t)\|_{\mathbb{L}^2}^2 dt \leq C. \end{aligned} \quad (3.14)$$

Since  $\mathbf{e}_i \in \mathbb{L}^2$  for  $i = 1, \dots, n$ , we see from Lemma 3.3 that

$$\int_0^t \int_D \left| (1 + \mu|\mathbf{u}_n(s)|^2)\mathbf{u}_n(s) \cdot \mathbf{e}_i \right| d\mathbf{x} ds \leq t^{1/2} \|\mathbf{e}_i\|_{\mathbb{L}^2} \|(1 + \mu|\mathbf{u}_n|^2)\mathbf{u}_n\|_{L^2(0,T;\mathbb{L}^2)} < \infty,$$

and thus from Fubini's theorem there holds

$$\int_0^t \Pi_n \left( (1 + \mu|\mathbf{u}_n(s)|^2)\mathbf{u}_n(s) \right) ds = \Pi_n \left( \int_0^t (1 + \mu|\mathbf{u}_n(s)|^2)\mathbf{u}_n(s) ds \right). \quad (3.15)$$

By using (3.15) and (3.2), the Minkowski and Hölder inequalities, and the second result of Lemma 3.3, we infer that

$$\begin{aligned}
\int_0^T \|\mathbf{B}_{n,3}(t)\|_{\mathbb{L}^2}^2 dt &= \int_0^T \left\| \int_0^t \Pi_n((1 + \mu|\mathbf{u}_n|^2(s))\mathbf{u}_n(s)) ds \right\|_{\mathbb{L}^2}^2 dt \\
&= \int_0^T \left\| \Pi_n\left(\int_0^t (1 + \mu|\mathbf{u}_n|^2(s))\mathbf{u}_n(s) ds\right) \right\|_{\mathbb{L}^2}^2 dt \\
&\leq \int_0^T \left\| \int_0^t (1 + \mu|\mathbf{u}_n|^2(s))\mathbf{u}_n(s) ds \right\|_{\mathbb{L}^2}^2 dt \\
&\leq \int_0^T \left( \int_0^t \|(1 + \mu|\mathbf{u}_n|^2(s))\mathbf{u}_n(s)\|_{\mathbb{L}^2} ds \right)^2 dt \\
&\leq \int_0^T t \int_0^t \|(1 + \mu|\mathbf{u}_n|^2(s))\mathbf{u}_n(s)\|_{\mathbb{L}^2}^2 ds dt \\
&\leq \int_0^T tC dt = CT^2.
\end{aligned} \tag{3.16}$$

Thus, (3.8) follows from (3.14) and (3.16).

Proof of (3.9): From Lemma 3.2,  $\Delta\mathbf{u}_n$  is uniformly bounded in  $L^2(0, T; \mathbb{L}^2)$ . By using the same arguments as in the proof of (3.8), we also deduce

$$\|\mathbf{B}_{n,1}\|_{H^1(0,T;\mathbb{L}^2)} \leq C. \tag{3.17}$$

Since  $\mathbb{L}^2 \hookrightarrow \mathbb{L}^{3/2}$  we see from (3.10) that  $\mathbb{L}^2 \hookrightarrow X^{-\beta}$  and thus  $H^1(0, T; \mathbb{L}^2) \hookrightarrow H^1(0, T; X^{-\beta})$ . It follows from (3.17) and (3.8) that  $\mathbf{B}_{n,1}$  and  $\mathbf{B}_{n,3}$  are uniformly bounded in  $H^1(0, T; X^{-\beta})$ . Together with (3.7) we have

$$\|\mathbf{u}_n\|_{H^1(0,T;X^{-\beta})} \leq C,$$

which complete the proof of this lemma.  $\square$

## 4 Existence of a weak solution

In this section, by using the method of compactness, we show that there is a subsequence of  $\{\mathbf{u}_n\}$  whose limit is a weak solution of (1.3).

Firstly, in the following lemma we prove the existence of a convergent subsequence of  $\mathbf{u}_n$  in a functional space.

**Lemma 4.1.** *Let  $D \subset \mathbb{R}^d$  be an open bounded domain with the  $C^m$  extension property and let  $\mathbf{u}_n$  be the solution of (3.3) for  $n = 1, 2, \dots$ . Assume that  $d < 2m$ , then there exist a subsequence of  $\{\mathbf{u}_n\}$  (still denoted by  $\{\mathbf{u}_n\}$ ) and  $\mathbf{u} \in C([0, T]; X^{-\bar{\beta}}) \cap L^{\bar{p}}(0, T; \mathbb{L}^4)$  such that*

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ strongly in } \mathbb{C}([0, T]; X^{-\bar{\beta}}) \cap L^{\bar{p}}(0, T; \mathbb{L}^4), \tag{4.1}$$

where  $\bar{\beta} > \frac{d}{6m}$  and  $\bar{p} \geq 4$ . Furthermore,

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; \mathbb{H}^1). \quad (4.2)$$

*Proof.* From (3.9), the sequence  $\{\mathbf{u}_n\}_n$  is uniformly bounded in  $H^1(0, T; X^{-\beta})$  with given  $\beta > \frac{d}{6m}$ . For each  $p \in [2, \infty)$ , thanks to Lemma 5.3 we have the continuous imbeddings

$$H^1(0, T; X^{-\beta}) \hookrightarrow \mathbb{W}^{\alpha, p}(0, T; X^{-\beta}) \quad \text{if } \alpha \in (0, \frac{1}{2}) \text{ and } \frac{1}{2} > \alpha - \frac{1}{p},$$

so by Lemma 3.2 the sequence  $\{\mathbf{u}_n\}_n$  is uniformly bounded in  $W^{\alpha, p}(0, T; X^{-\beta}) \cap L^p(0, T; \mathbb{H}^1)$ .

From [19, Theorem 1.4.8],  $X^\nu$  is compactly embedded in  $X^{\nu'}$  whenever  $\nu$  and  $\nu'$  are real numbers with  $\nu > \nu'$ . Since  $\mathbb{H}^1 = X^{1/2}$ , there exists  $\gamma \in [-\beta, \frac{1}{2})$  such that the embeddings

$$\mathbb{H}^1 \hookrightarrow X^\gamma \hookrightarrow X^{-\beta} \text{ are compact.}$$

By using Lemmas 5.4–5.5, we deduce the compact embeddings

$$W^{\alpha, p}(0, T; X^{-\beta}) \cap L^p(0, T; \mathbb{H}^1) \hookrightarrow L^p(0, T; X^\gamma), \quad (4.3)$$

$$\mathbb{W}^{\alpha, p}(0, T; X^{-\beta}) \hookrightarrow C([0, T]; X^{-\bar{\beta}}) \text{ if } \bar{\beta} > \beta \text{ and } \alpha p > 1. \quad (4.4)$$

From Lemma 5.2,  $X^\gamma$  is continuously embedded in  $\mathbb{L}^q$  when  $\gamma > \frac{d(q-2)}{2mq}$ , so

$$L^p(0, T; X^\gamma) \hookrightarrow L^p(0, T; \mathbb{L}^q) \quad \text{when } \gamma > \frac{d(q-2)}{2mq}. \quad (4.5)$$

It follows from (4.3), (4.4) and (4.5) that if

$$\bar{\beta} > \beta > \frac{d}{6m}, \quad \frac{1}{2} > \alpha - \frac{1}{p} > 0 \quad \text{and} \quad \frac{d(q-2)}{2mq} < \frac{1}{2}, \quad (4.6)$$

then the embedding

$$W^{\alpha, p}(0, T; X^{-\beta}) \cap L^p(0, T; \mathbb{H}^1) \hookrightarrow C([0, T]; X^{-\bar{\beta}}) \cap L^p(0, T; \mathbb{L}^q) \text{ is compact.}$$

In what follows, we choose  $p = \bar{p} \geq 4, q = 4, \bar{\beta} > \frac{d}{6m}$ . Thus, with the assumption  $d < 2m$  the condition (4.6) holds. It follows that there exist a subsequence of  $\{\mathbf{u}_n\}$  (still denoted by  $\{\mathbf{u}_n\}$ ) and  $\mathbf{u} \in C([0, T]; X^{-\bar{\beta}}) \cap L^{\bar{p}}(0, T; \mathbb{L}^4)$  such that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ strongly in } \mathbb{C}([0, T]; X^{-\bar{\beta}}) \cap L^{\bar{p}}(0, T; \mathbb{L}^4).$$

Furthermore, from Lemma 3.2, the sequence  $\{\mathbf{u}_n\}_n$  is uniformly bounded in  $L^2(0, T; \mathbb{H}^1)$ . Thus, there exists a subsequence of  $\{\mathbf{u}_n\}$  (still denoted by  $\{\mathbf{u}_n\}$ ) such that

$$\mathbf{u}_n \rightarrow \mathbf{u} \text{ weakly in } L^2(0, T; \mathbb{H}^1),$$

which completes the proof of this lemma.  $\square$

In the remaining part of this paper, we will choose  $\bar{p} = 8$  in Lemma 4.1.

Secondly, we find the limits of sequences  $\{\Pi_n(\mathbf{u}_n \times \Delta \mathbf{u}_n)\}_n$  and  $\{\Pi_n((1 + |\mathbf{u}_n|^2)\mathbf{u}_n)\}_n$  and their relationship with  $\mathbf{u}$  in the following lemmas.

Since the Banach spaces  $\mathbb{L}^2(0, T; \mathbb{L}^{3/2})$  and  $\mathbb{L}^2(0, T; X^{-\beta})$  are all reflexive, from Lemmas 3.3–3.6 and by the Banach-Alaoglu Theorem there exist subsequences of  $\{\mathbf{u}_n \times \Delta \mathbf{u}_n\}$  and of  $\{\Pi_n(\mathbf{u}_n \times \Delta \mathbf{u}_n)\}$  (still denoted by  $\{\mathbf{u}_n \times \Delta \mathbf{u}_n\}$ ,  $\{\Pi_n(\mathbf{u}_n \times \Delta \mathbf{u}_n)\}$ , respectively); and  $Z \in \mathbb{L}^2(0, T; \mathbb{L}^{3/2})$ ,  $\bar{Z} \in \mathbb{L}^2(0, T; X^{-\beta})$  such that

$$\mathbf{u}_n \times \Delta \mathbf{u}_n \rightarrow Z \text{ weakly in } \mathbb{L}^2(0, T; \mathbb{L}^{3/2}) \quad (4.7)$$

$$\Pi_n(\mathbf{u}_n \times \Delta \mathbf{u}_n) \rightarrow \bar{Z} \text{ weakly in } \mathbb{L}^2(0, T; X^{-\beta}). \quad (4.8)$$

**Lemma 4.2.** *If  $Z$  and  $\bar{Z}$  defined as above, then  $Z = \bar{Z}$  in  $\mathbb{L}^2(0, T; X^{-\beta})$ .*

*Proof.* From (3.10), we infer that  $Z \in \mathbb{L}^2(0, T; X^{-\beta})$ . For every  $n \in \mathbb{N}$ , let us denote  $X_n^\beta := \{\Pi_n \mathbf{x} : \mathbf{x} \in X^\beta\} = S_n$  with the norm inherited from  $X^\beta$ . Then from Lemma 5.1,  $\cup_{n=1}^\infty X_n^\beta$  is dense  $X^\beta$  and thus  $\cup_{n=1}^\infty \mathbb{L}^2(0, T; X_n^\beta)$  is dense  $\mathbb{L}^2(0, T; X^\beta)$ . Hence, it is sufficient to prove that for any  $\phi_m \in \mathbb{L}^2(0, T; X_m^\beta)$ ,

$$\mathbb{L}^2(0, T; X^{-\beta}) \langle \bar{Z}, \phi_m \rangle_{\mathbb{L}^2(0, T; X^\beta)} = \mathbb{L}^2(0, T; X^{-\beta}) \langle Z, \phi_m \rangle_{\mathbb{L}^2(0, T; X^\beta)}.$$

For this aim let us fix  $m \in \mathbb{N}$  and  $\phi_m \in \mathbb{L}^2(0, T; X_m^\beta)$ . Since  $X_m^\beta \subset X_n^\beta$  for any  $n \geq m$ , we have

$$\begin{aligned} \mathbb{L}^2(0, T; X^{-\beta}) \langle \Pi_n(\mathbf{u}_n \times \Delta \mathbf{u}_n), \phi_m \rangle_{\mathbb{L}^2(0, T; X^\beta)} &= \int_0^T \langle \Pi_n(\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)), \phi_m \rangle_{X^\beta} dt \\ &= \int_0^T \langle \Pi_n(\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)), \phi_m \rangle_{\mathbb{L}^2} dt \\ &= \int_0^T \langle (\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)), \Pi_n \phi_m \rangle_{\mathbb{L}^2} dt \\ &= \int_0^T \langle (\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)), \phi_m \rangle_{\mathbb{L}^2} dt \\ &= \mathbb{L}^2(0, T; X^{-\beta}) \langle (\mathbf{u}_n \times \Delta \mathbf{u}_n), \phi_m \rangle_{\mathbb{L}^2(0, T; X^\beta)}. \end{aligned}$$

Hence the result follows by taking the limit as  $n$  tends to infinity of the above equation and using (4.7)–(4.8).  $\square$

**Lemma 4.3.** *For any  $\phi \in \mathbb{W}^{1,4}(D) \cap X^\beta$ , there holds*

$$\lim_{n \rightarrow \infty} \int_0^T \langle \Pi_n(\mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t)), \phi \rangle_{X^\beta} dt = - \int_0^T \langle \mathbf{u}(t) \times \nabla \mathbf{u}(t), \nabla \phi \rangle_{\mathbb{L}^2} dt \quad (4.9)$$

$$\text{and} \quad \lim_{n \rightarrow \infty} \int_0^T \langle \Pi_n((1 + \mu |\mathbf{u}_n|^2(t)) \mathbf{u}_n(t)), \phi \rangle_{\mathbb{L}^2} dt = \int_0^T \langle (1 + \mu |\mathbf{u}|^2(t)) \mathbf{u}(t), \phi \rangle_{\mathbb{L}^2} dt \quad (4.10)$$

*Proof.* Proof of (4.9): From (4.7)–(4.8), Lemma 4.2, and

$$\langle \mathbf{u}_n(t) \times \Delta \mathbf{u}_n(t), \phi \rangle_{\mathbb{L}^2} = -\langle \mathbf{u}_n(t) \times \nabla \mathbf{u}_n(t), \nabla \phi \rangle_{\mathbb{L}^2},$$

it is sufficient to prove that

$$\lim_{n \rightarrow \infty} \int_0^T \langle \mathbf{u}_n(t) \times \nabla \mathbf{u}_n(t), \nabla \phi \rangle_{\mathbb{L}^2} dt = \int_0^T \langle \mathbf{u}(t) \times \nabla \mathbf{u}(t), \nabla \phi \rangle_{\mathbb{L}^2} dt. \quad (4.11)$$

By using the triangle and Hölder inequalities together with Lemma 3.2, we see that

$$\begin{aligned} & \left| \int_0^T \langle \mathbf{u}_n(t) \times \nabla \mathbf{u}_n(t), \nabla \phi \rangle_{\mathbb{L}^2} dt - \int_0^T \langle \mathbf{u}(t) \times \nabla \mathbf{u}(t), \nabla \phi \rangle_{\mathbb{L}^2} dt \right| \\ & \leq \left| \int_0^T \langle (\mathbf{u}_n(t) - \mathbf{u}(t)) \times \nabla \mathbf{u}_n(t), \nabla \phi \rangle_{\mathbb{L}^2} dt \right| \\ & \quad + \left| \int_0^T \langle \mathbf{u}(t) \times (\nabla \mathbf{u}_n(t) - \nabla \mathbf{u}(t)), \nabla \phi \rangle_{\mathbb{L}^2} dt \right| \\ & \leq \|\mathbf{u}_n - \mathbf{u}\|_{L^4(0,T;\mathbb{L}^4)} \|\nabla \mathbf{u}_n\|_{L^2(0,T;\mathbb{L}^2)} \|\nabla \phi\|_{L^4(0,T;\mathbb{L}^4)} \\ & \quad + \left| \int_0^T \langle \nabla \mathbf{u}_n(t) - \nabla \mathbf{u}(t), \nabla \phi \times \mathbf{u}(t) \rangle_{\mathbb{L}^2} dt \right| \\ & \leq C \|\mathbf{u}_n - \mathbf{u}\|_{L^4(0,T;\mathbb{L}^4)} + \left| \int_0^T \langle \nabla \mathbf{u}_n(t) - \nabla \mathbf{u}(t), \nabla \phi \times \mathbf{u}(t) \rangle_{\mathbb{L}^2} dt \right|. \end{aligned}$$

Hence, (4.9) follows by passing to the limit as  $n$  tends to infinity of the above inequality and using (4.1)–(4.2), noting that  $\nabla \phi \times \mathbf{u} \in L^2(0,T;\mathbb{L}^2)$  since  $\mathbf{u} \in L^4(0,T;\mathbb{L}^4)$ .

Proof of (4.10): Since  $\Pi_n$  is a self-adjoint operator on  $\mathbb{L}^2$ , we have

$$\langle \Pi_n((1 + \mu|\mathbf{u}_n|^2(t))\mathbf{u}_n(t)), \phi \rangle_{\mathbb{L}^2} = \langle \mathbf{u}_n, \phi \rangle_{\mathbb{L}^2} + \mu \langle |\mathbf{u}_n|^2(t)\mathbf{u}_n(t), \Pi_n \phi \rangle_{\mathbb{L}^2},$$

so from (4.2), it is sufficient to prove that

$$\lim_{n \rightarrow \infty} \int_0^T \langle |\mathbf{u}_n|^2(t)\mathbf{u}_n(t), \Pi_n \phi \rangle_{\mathbb{L}^2} dt = \int_0^T \langle |\mathbf{u}|^2(t)\mathbf{u}(t), \phi \rangle_{\mathbb{L}^2} dt.$$

By using the triangle and Hölder inequalities, (3.4) and Lemma 3.2, we see that

$$\begin{aligned}
& \left| \int_0^T \langle |\mathbf{u}_n|^2(t) \mathbf{u}_n(t), \Pi_n \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt - \int_0^T \langle |\mathbf{u}|^2(t) \mathbf{u}(t), \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt \right| \\
& \leq \left| \int_0^T \langle |\mathbf{u}_n|^2(t) \mathbf{u}_n(t), \Pi_n \boldsymbol{\phi} - \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt \right| + \left| \int_0^T \langle |\mathbf{u}_n|^2(t) (\mathbf{u}_n(t) - \mathbf{u}(t)), \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt \right| \\
& \quad + \left| \int_0^T \langle (|\mathbf{u}_n|^2(t) - |\mathbf{u}|^2(t)) \mathbf{u}(t), \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt \right| \\
& \leq \|\Pi_n \boldsymbol{\phi} - \boldsymbol{\phi}\|_{\mathbb{L}^2} \int_0^T \|\mathbf{u}_n^3(t)\|_{\mathbb{L}^2} dt \\
& \quad + \| |\mathbf{u}_n|^2 \|_{L^2(0,T;\mathbb{L}^2)} \|\mathbf{u}_n - \mathbf{u}\|_{L^4(0,T;\mathbb{L}^4)} \|\boldsymbol{\phi}\|_{L^4(0,T;\mathbb{L}^4)} \\
& \quad + \|\mathbf{u}_n - \mathbf{u}\|_{L^4(0,T;\mathbb{L}^4)} \|\mathbf{u}_n + \mathbf{u}\|_{L^4(0,T;\mathbb{L}^4)} \|\mathbf{u}\|_{L^4(0,T;\mathbb{L}^4)} \|\boldsymbol{\phi}\|_{L^4(0,T;\mathbb{L}^4)} \\
& \leq C \|\Pi_n \boldsymbol{\phi} - \boldsymbol{\phi}\|_{\mathbb{L}^2} + C \|\mathbf{u}_n - \mathbf{u}\|_{L^4(0,T;\mathbb{L}^4)}.
\end{aligned}$$

Hence, (4.10) follows by passing to the limit as  $n$  tends to infinity of the above inequality and using (4.1).  $\square$

We wish to use the notations  $\Delta \mathbf{u}$  and  $\mathbf{u} \times \Delta \mathbf{u}$  in the equation satisfied by  $\mathbf{u}$ . These notations are defined as follow.

From Lemma 3.2, we have  $\Delta \mathbf{u}_n$  is uniformly bounded in  $L^2(0, T; \mathbb{L}^2)$ . Thus, there exist a subsequence of  $\{\Delta \mathbf{u}_n\}$  (still denoted by  $\{\Delta \mathbf{u}_n\}$ ) and  $\mathbf{Y} \in L^2(0, T; \mathbb{L}^2)$  such that

$$\Delta \mathbf{u}_n \rightharpoonup \mathbf{Y} \text{ weakly in } L^2(0, T; \mathbb{L}^2).$$

Together with (4.2) we obtain

$$\int_0^T \langle \mathbf{Y}(t), \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt = - \int_0^T \langle \nabla \mathbf{u}(t), \nabla \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt,$$

for  $\boldsymbol{\phi} \in \mathbb{W}^{1,4}(D) \cap X^\beta$ .

**Notation 4.1.** By denoting  $\Delta \mathbf{u} := \mathbf{Y}$ , we have  $\Delta \mathbf{u} \in L^2(0, T; \mathbb{L}^2)$ .

From (4.7) and (4.11), for  $\boldsymbol{\phi} \in \mathbb{W}^{1,4}(D) \cap X^\beta$  we have

$$\int_0^T \langle \mathbf{Z}(t), \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt = - \int_0^T \langle \mathbf{u}(t) \times \nabla \mathbf{u}(t), \nabla \boldsymbol{\phi} \rangle_{\mathbb{L}^2} dt.$$

**Notation 4.2.** By denoting  $\mathbf{u} \times \Delta \mathbf{u} := \mathbf{Z}$ , we have  $\mathbf{u} \times \Delta \mathbf{u} \in L^2(0, T; \mathbb{L}^{3/2})$ .

We now ready to prove the main theorem.

*Proof.* Proof of theorem 2.2

For any test function  $\phi \in \mathbb{W}^{1,4}(D) \cap X^\beta$ , from (3.5) and integrating by parts, we have

$$\begin{aligned} \langle \mathbf{u}_n(t), \phi \rangle_{\mathbb{L}^2} &= \langle \mathbf{u}_n(0), \phi \rangle_{\mathbb{L}^2} - \kappa_1 \int_0^t \langle \nabla \mathbf{u}_n(s), \nabla \phi \rangle_{\mathbb{L}^2} ds + \gamma \int_0^t \langle \Pi_n(\mathbf{u}_n(s) \times \Delta \mathbf{u}_n(s)), \phi \rangle_{\mathbb{L}^2} ds \\ &\quad - \kappa_2 \int_0^t \langle \Pi_n((1 + \mu |\mathbf{u}_n|^2(s)) \mathbf{u}_n(s)), \phi \rangle_{\mathbb{L}^2} ds. \end{aligned}$$

By passing to the limit as  $n$  tends to infinity of the above equation and using (4.1)–(4.2) and Lemma 4.3, we obtain that  $\mathbf{u}$  satisfies (2.1).

Furthermore, using Notations 4.1–4.2, we infer that  $\mathbf{u}$  satisfies the following equation in  $X^{-\beta}$  with  $\beta > \frac{4+d}{4m}$ ,

$$\mathbf{u}(t) = \mathbf{u}_0 + \kappa_1 \int_0^t \Delta \mathbf{u}(s) ds + \gamma \int_0^t \mathbf{u}(s) \times \Delta \mathbf{u}(s) ds - \kappa_2 \int_0^t (1 + |\mathbf{u}|^2(s)) \mathbf{u}(s) ds. \quad (4.12)$$

Proof of (a): It is enough to prove that the terms in equation (4.12) are in the space  $\mathbb{L}^{3/2}$ . Since we wish to use the following arguments in the proof of (b), we will use  $\int_\tau^t$  for  $\tau \in [0, t]$  instead of just  $\int_0^t$ . By using the Minkowski inequality and the continuous embedding  $\mathbb{L}^2 \hookrightarrow \mathbb{L}^{3/2}$ , we have

$$\left\| \int_\tau^t \Delta \mathbf{u}(s) ds \right\|_{\mathbb{L}^{3/2}} \leq \int_\tau^t \|\Delta \mathbf{u}(s)\|_{\mathbb{L}^{3/2}} ds \leq C \int_\tau^t \|\Delta \mathbf{u}(s)\|_{\mathbb{L}^2} ds \leq C(t - \tau)^{\frac{1}{2}}, \quad (4.13)$$

and

$$\begin{aligned} \left\| \int_\tau^t \mathbf{u}(s) \times \Delta \mathbf{u}(s) ds \right\|_{\mathbb{L}^{3/2}} &\leq \int_\tau^t \|\mathbf{u}(s) \times \Delta \mathbf{u}(s)\|_{\mathbb{L}^{3/2}} ds \\ &\leq (t - \tau)^{\frac{1}{2}} \left( \int_\tau^t \|\mathbf{u}(s) \times \Delta \mathbf{u}(s)\|_{\mathbb{L}^{3/2}}^2 ds \right)^{\frac{1}{2}} \leq C(t - \tau)^{\frac{1}{2}}. \end{aligned} \quad (4.14)$$

For the last term in (4.12), it is sufficient to prove that  $\left\| \int_0^t |\mathbf{u}|^2(s) \mathbf{u}(s) ds \right\|_{\mathbb{L}^{3/2}} < \infty$ . Indeed, by using the Hölder and Minkowski inequalities we deduce

$$\begin{aligned} \left\| \int_\tau^t |\mathbf{u}|^2(s) \mathbf{u}(s) ds \right\|_{\mathbb{L}^{3/2}}^{3/2} &= \int_D \left| \int_\tau^t |\mathbf{u}|^2(s) \mathbf{u}(s) ds \right|^{3/2} d\mathbf{x} \\ &\leq \int_D \left( \int_\tau^t |\mathbf{u}|^4(s) ds \right)^{\frac{3}{4}} \left( \int_\tau^t |\mathbf{u}|^2(s) ds \right)^{\frac{3}{4}} d\mathbf{x} \\ &\leq \left( \int_D \int_\tau^t |\mathbf{u}|^4(s) ds d\mathbf{x} \right)^{\frac{3}{4}} \left( \int_D \left( \int_\tau^t |\mathbf{u}|^2(s) ds \right)^3 d\mathbf{x} \right)^{\frac{1}{4}} \\ &\leq (t - \tau)^{\frac{3}{8}} \|\mathbf{u}\|_{L^8(0,T;\mathbb{L}^4)}^3 \|\mathbf{u}\|_{L^2(0,T;\mathbb{L}^6)}^{3/2} \leq C(t - \tau)^{\frac{3}{8}}, \end{aligned} \quad (4.15)$$

where the last inequality follows because the fact that

$$\mathbf{u} \in L^8(0, T; \mathbb{L}^4) \cap L^2(0, T; \mathbb{L}^6),$$

which is a consequence of Lemma 4.1 and the embedding

$$L^2(0, T; \mathbb{H}^1) \hookrightarrow L^2(0, T; \mathbb{L}^6).$$

By taking  $\tau = 0$  in (4.13)–(4.15), we infer that  $\mathbf{u}$  satisfies (4.12) in  $\mathbb{L}^{3/2}$ .

Proof of (b): From (4.13)–(4.15), we obtain

$$\sup_{0 \leq \tau < t \leq T} \frac{\|\mathbf{u}(t) - \mathbf{u}(\tau)\|_{\mathbb{L}^{3/2}}}{|t - \tau|^{1/4}} < \infty;$$

it follows that  $\mathbf{u} \in C^{\bar{\alpha}}([0, T]; \mathbb{L}^{3/2})$  for every  $\bar{\alpha} \in (0, \frac{1}{4}]$ .

Proof of (c): Finally, property (c) follows from applying weak lower semicontinuity of norms in the first inequality of Lemma 3.2, which complete the proof of our main theorem.  $\square$

## 5 Appendix

**Lemma 5.1.** *Let  $X_n^\beta := \{\Pi_n \mathbf{x} : \mathbf{x} \in X^\beta\}$  with the norm inherited from  $X^\beta$ . Then*

$$\lim_{n \rightarrow \infty} \|\Pi_n \mathbf{x} - \mathbf{x}\|_{X^\beta} = 0 \quad \text{for every } \mathbf{x} \in X^\beta.$$

*Proof.* For  $\mathbf{x} \in X^\beta$ , we have  $\Pi_n \mathbf{x} = \sum_{i=1}^n \langle \mathbf{x}, \mathbf{e}_i \rangle_{\mathbb{L}^2} \mathbf{e}_i$ , thus  $\mathbf{x} - \Pi_n \mathbf{x} = \sum_{i=n+1}^\infty \langle \mathbf{x}, \mathbf{e}_i \rangle_{\mathbb{L}^2} \mathbf{e}_i$ . By using orthonormal property of  $\{\mathbf{e}_i\}$ , we obtain

$$\lim_{n \rightarrow \infty} \|\Pi_n \mathbf{x} - \mathbf{x}\|_{X^\beta} = \lim_{n \rightarrow \infty} \sum_{i=n+1}^\infty (1 + \lambda_i)^\beta \langle \mathbf{x}, \mathbf{e}_i \rangle_{\mathbb{L}^2}^2 = 0,$$

as  $\|\mathbf{x}\|_{X^\beta} := \sum_{i=1}^\infty (1 + \lambda_i)^\beta \langle \mathbf{x}, \mathbf{e}_i \rangle_{\mathbb{L}^2}^2 < \infty$ .  $\square$

For the reader's convenience we will recall some embedding results that are crucial for the proof of convergence of the approximating sequence  $\{\mathbf{u}_n\}$ .

**Lemma 5.2.** *[19, Theorem 1.6.1] Suppose  $\Omega \subset \mathbb{R}^d$  is an open set having the  $C^m$  extension property,  $1 \leq p < \infty$  and  $A$  is a sectorial operator in  $X = \mathbb{L}^p(\Omega)$  with  $D(A) = X^1 \hookrightarrow \mathbb{W}^{m,p}(\Omega)$  for some  $m \geq 1$ . Then for  $0 \leq \beta \leq 1$ ,*

$$X^\beta \hookrightarrow \mathbb{W}^{k,q}(\Omega) \quad \text{when} \quad k - \frac{d}{q} < m\beta - \frac{d}{p}, \quad q \geq p,$$

and  $X^\beta \hookrightarrow \mathbb{C}^\alpha(\Omega) \quad \text{when} \quad 0 \leq \alpha < m\beta - \frac{d}{p}.$



**Lemma 5.3.** [22, Corollary 19] Suppose  $s \geq r$ ,  $p \leq q$  and  $s - 1/p \geq r - 1/q$  ( $0 < r \leq s < 1$ ,  $1 \leq p \leq q \leq \infty$ ). Let  $E$  be a Banach space and  $I$  be an interval of  $\mathbb{R}$ . Then

$$W^{s,p}(I; E) \hookrightarrow W^{r,q}(I; E).$$

**Lemma 5.4.** [12, Theorem 2.1] Assume that  $B_0 \subset B \subset B_1$  are Banach spaces,  $B_0$  and  $B_1$  reflexive with compact embedding of  $B_0$  in  $B$ . Let  $p \in (1, \infty)$  and  $\alpha \in (0, 1)$  be given. Then the embedding

$$L^p(0, T; B_0) \cap W^{\alpha,q}(0, T; B_1) \hookrightarrow L^p(0, T; B) \text{ is compact.}$$

**Lemma 5.5.** [12, Theorem 2.2] Assume that  $B_0 \subset B$  are Banach spaces such that the embedding  $B_0 \hookrightarrow B$  is compact. Let  $p \in (1, \infty)$  and  $0 < \alpha < 1$  and  $\alpha p > 1$ . Then the embedding

$$W^{\alpha,q}(0, T; B_0) \hookrightarrow C([0, T]; B) \text{ is compact.}$$

## References

- [1] F. Alouges and A. Soyeur. On global weak solutions for Landau-Lifshitz equations: Existence and nonuniqueness. *Nonlinear Anal.*, **18** (1992), 1071–1084.
- [2] U. Atxitia, O. Chubykalo-Fesenko, N. Kazantseva, D. Hinzke, U. Nowak, and R. W. Chantrell. Micromagnetic modeling of laser-induced magnetization dynamics using the Landau-Lifshitz-Bloch equation. *Applied Physics Letters*, **91** (2007).
- [3] Ľ. Bañas, Z. Brzeźniak, M. Neklyudov, and A. Prohl. A convergent finite-element-based discretization of the stochastic Landau–Lifshitz–Gilbert equation. *IMA Journal of Numerical Analysis*, (2013).
- [4] Ľ. Bañas, Z. Brzeźniak, M. Neklyudov, and A. Prohl. *Stochastic Ferromagnetism—Analysis and Numerics*. de Gruyter Series in Mathematics, 58. de Gruyter, 2013.
- [5] Ľ. Bañas, Z. Brzeźniak, and A. Prohl. Computational studies for the stochastic Landau–Lifshitz–Gilbert equation. *SIAM J. Sci. Comput.*, **35** (2013), B62–B81.
- [6] Z. Brzeźniak, B. Goldys, and T. Jegaraj. Weak solutions of a stochastic Landau–Lifshitz–Gilbert equation. *Applied Mathematics Research eXpress*, (2012), 1–33.
- [7] Z. Brzeźniak and L. Li. Weak solutions of the stochastic Landau–Lifshitz–Gilbert equation with non-zero anisotropy energy. *Applied Mathematics Research eXpress*, (2016).

- [8] G. Carbou and P. Fabrie. Regular solutions for Landau–Lifschitz equation in a bounded domain. *Differential Integral Equations*, **14** (2001), 213–229.
- [9] I. Cimrák. A survey on the numerics and computations for the Landau-Lifshitz equation of micromagnetism. *Arch. Comput. Methods Eng.*, **15** (2008), 277–309.
- [10] L. C. Evans. *Partial Differential Equations*. American Mathematical Society, Berlin, 2 edition, 1998.
- [11] R. F. L. Evans, D. Hinze, U. Atxitia, U. Nowak, R. W. Chantrell, and O. Chubykalo-Fesenko. Stochastic form of the Landau-Lifshitz-Bloch equation. *Phys. Rev. B*, **85** (2012), 014433.
- [12] F. Flandoli and D. Gatarek. Martingale and stationary solutions for stochastic Navier–Stokes equations. *Probability Theory and Related Fields*, **102** (1995), 367–391.
- [13] A. Friedman. *Partial Differential Equations*. New York, 1969.
- [14] D. Garanin. Generalized equation of motion for a ferromagnet. *Physica A: Statistical Mechanics and its Applications*, **172** (1991), 470 – 491.
- [15] D. A. Garanin. Fokker-Planck and Landau-Lifshitz-Bloch equations for classical ferromagnets. *Phys. Rev. B*, **55** (1997), 3050–3057.
- [16] T. Gilbert. A Lagrangian formulation of the gyromagnetic equation of the magnetic field. *Phys Rev*, **100** (1955), 1243–1255.
- [17] B. Goldys, K.-N. Le, and T. Tran. A finite element approximation for the stochastic Landau–Lifshitz–Gilbert equation. *Journal of Differential Equations*, **260** (2016), 937 – 970.
- [18] B. Guo and S. Ding. *Landau–Lifshitz Equations*, volume 1 of *Frontiers of Research with the Chinese Academy of Sciences*. World Scientific Publishing Co. Pty. Ltd., Hackensack, NJ, 2008.
- [19] D. Henry. *Geometric Theory of Semilinear Parabolic Equations*. Springer-Verlag Berlin and Heidelberg, New York, 1981.
- [20] M. Kružík and A. Prohl. Recent developments in the modeling, analysis, and numerics of ferromagnetism. *SIAM Rev.*, **48** (2006), 439–483.
- [21] L. Landau and E. Lifshitz. On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. *Phys Z Sowjetunion*, **8** (1935), 153–168.
- [22] J. Simon. Sobolev, Besov and Nikolskii fractional spaces: Imbeddings and comparisons for vector valued spaces on an interval. *Annali di Matematica Pura ed Applicata*, **157** (1990), 117–148.